A Unified Primal-Dual Algorithm Framework Based on Bregman Iteration

Abstract: I will present a primal-dual algorithm of a class of convex optimization problems that arise from various signal and image processing applications. I will explain connections between existing algorithms (in particular Bregman Iteration) and the primal-dual algorithm. Furthermore, the convergence analysis of the proposed algorithm will be given in detail.

Keywords: Saddle point, Bregman Iteration, Uzawa method, Proximal point iteration, Convergence analysis

Outline: Consider the following optimization problem:

\[
\begin{align*}
\min_{x,z} & \quad J(z) + H(x) \\
\text{subject to} & \quad Bx = z
\end{align*}
\]

where \( J(.) \) and \( H(.) \) are convex functions, \( x \in \mathbb{R}^N, z \in \mathbb{R}^K \) and \( B \) is an \( N \times K \) matrix.

The primal-dual algorithm for solving this problem is

\[
\begin{align*}
x^{k+1} &= \arg\min_x (H(x) + \langle y^k | Bx \rangle + \frac{a}{2} \| Bx - z^k \|^2 + \frac{1}{2} \| x - x^k \|^2_{Q_1}) \\
z^{k+1} &= \arg\min_z (J(z) - \langle y^k | z \rangle + \frac{\alpha}{2} \| Bx^{k+1} - z \|^2 + \frac{1}{2} \| z - z^k \|^2_{Q_2}) \\
y^{k+1} &= y^k + (Bx^{k+1} - z^{k+1})
\end{align*}
\]

where \( Q_1, C \) are positive definite matrices and \( Q_2 \) is a positive semi-definite matrix.

We will discuss the convergence analysis of this algorithm, using the following Theorem.

Theorem: Assume \( Q_1, C \) are positive definite matrices and \( Q_2 \) is positive semi-definite and

\[ 0 < \frac{1}{\lambda_m^C} \leq \alpha \quad (\lambda_m^C \text{ is the smallest eigenvalue of the matrix } C) \]. Then

\[
\begin{align*}
\| Bx^k - z^k \| &\to 0, \\
J(z^k) &\to J(\bar{z}), \\
H(z^k) &\to H(\bar{z})
\end{align*}
\]

for the sequence \((x^k, z^k, y^k)\) which is generated by the primal-dual algorithm. Also the limit point of \((x^k, z^k, y^k)\) is a saddle point of \( L(x, z; y) \).