A Unified Primal-Dual Algorithm Framework Based on Bregman Iteration

Abstract: I will present a primal-dual algorithm of a class of convex optimization problems that arise from various signal and image processing applications.
I will explain connections between existing algorithms (in particular Bregman Iteration) and the primal-dual algorithm. Furthermore, the convergence analysis of the proposed algorithm will be given in detail.

Keywords: Saddle point, Bregman Iteration, Uzawa method, Proximal point iteration, Convergence analysis

Outline: Consider the following optimization problem:

$$\min_{x,z} \quad J(z) + H(x)$$

$$\text{subject to} \quad Bx = z$$

where $J(.)$ and $H(.)$ are convex functions, $x \in R^N, z \in R^K$ and B is an N by K matrix.

The *primal-dual algorithm* for solving this problem is

$$\begin{cases} x^{k+1} = arg\min_x (H(x) + \langle y^k | Bx \rangle + \frac{\alpha}{2}\left\|Bx - z^k\right\|^2 + \frac{1}{2}\|x - x^k\|_{Q_1}^2) \\ z^{k+1} = arg\min_z (J(z) - \langle y^k | z \rangle + \frac{\alpha}{2}\left\|Bx^{k+1} - z\right\|^2 + \frac{1}{2}\left\|z - z^k\right\|_{Q_2}^2) \\ Cy^{k+1} = Cy^k + \left(Bx^{k+1} - z^{k+1}\right) \end{cases}$$

where Q_1, C are positive definite matrices and Q_2 is a positive semi-definite matrix.

We will discuss the convergence analysis of this algorithm, using the following Theorem.

Theorem: Assume Q_1, C are positive definite matrices and Q_2 is positive semi-definite and

$0 < \frac{1}{\lambda_m^C} \leq \alpha$ (λ_m^C is the smallest eigenvalue of the matrix C). Then

$$\left\|Bx^k - z^k\right\| \to 0,$$

$$J\left(z^k\right) \to J(\bar{z}),$$

$$H\left(z^k\right) \to H(\bar{z})$$

for the sequence (x^k, z^k, y^k) which is generated by the primal-dual algorithm. Also the limit point of (x^k, z^k, y^k) is a saddle point of $L(x, z; y)$.