Recovering small and extended corroded regions in EIT

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Introduction to the Problem

G. Granados and I. Harris, Reconstruction of small and extended regions in EIT with a Robin transmission condition. *Inverse Problems*, **38** 105009 (2022) (arXiv:2203.09551).

Inverse Boundary Value Problems

$$\text{Input} \Longrightarrow \boxed{\text{Physical Model}} \Longrightarrow \text{Output}$$

Input: Boundary Data

Physical Model: Partial Differential Equation

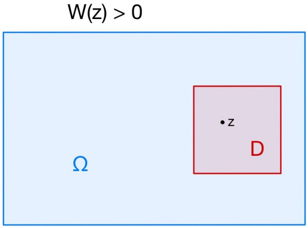
Output: Values of the Solution

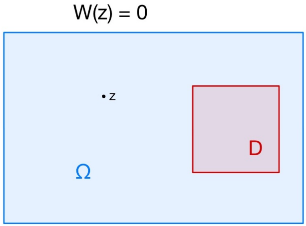
Forward Problem: Input + Physical Model = Output

Inverse Problem: Output + Input = Physical Model

Reconstruction via Qualitative Methods

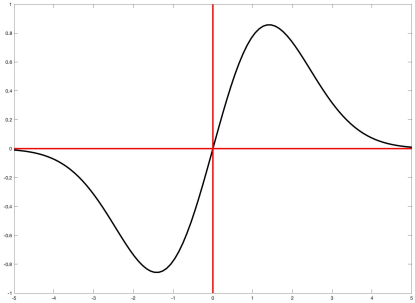
Qualitative Methods: Using nonlinear model to reconstruct limited information with **little a priori** information, such as the support of the region in a computationally simple and analytically rigorous manner.





Restrictions for Iterative Methods

Consider finding the root of $x\exp(-x^2/4) = 0$.



Iteration n	0	10	20	30	40
Sequence x_n	3	7.4236	9.8408	11.7439	13.3680

Table: Newton's Method for solving the equation $x\exp(-x^2/4) = 0$.

Restriction of Iterative Methods

Good initial guest is usually needed to insure Convergence.

This requires **a priori** information that may not be known:

- The number of objects to be recovered.
- Knowledge of the type of Boundary Condition.
- Estimates for the physical parameters/coefficients.

“A lack of information cannot be remedied by any mathematical trickery”- C. Lanczos

Electrical Impedance Tomography

Electrical impedance tomography (EIT) is a nondestructive type of imaging method in which the physical parameters of a material is recovered from surface electrode measurements.

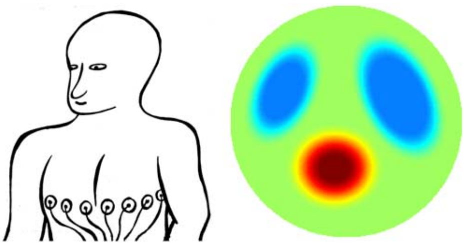
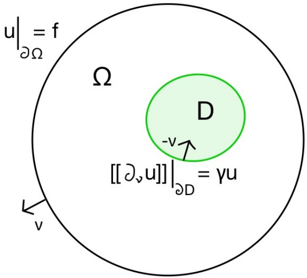


Figure: Picture taken from http://www.siltanen-research.net/

The EIT Problem under consideration

The electrostatic potential function $u \in H^1(\Omega)$ is Harmonic in $\Omega \backslash \partial D$ for

any prescribed voltage $f \in H^{1/2}(\partial\Omega)$.



Assumption and Notation

Let $\Omega \subset \mathbb{R}^d$ is a simply connected open set with $\partial\Omega$ Lipschitz.

Here $D \subset \Omega$ is a (possibly multiple) connected open set such that

- The boundary ∂D is $\mathcal{C}^2$.
- We have that $\text{dist}(\partial\Omega, \overline{D}) > 0$.

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- The boundary ∂D is $\mathcal{C}^2$.
- We have that $\text{dist}(\partial\Omega, \overline{D}) > 0$.

We define

$$[\![\partial_\nu u]\!]\big|_{\partial D} := (\partial_\nu u^+ - \partial_\nu u^-)\big|_{\partial D}.$$

The '$+$' notation represents the trace taken from $\Omega \setminus \overline{D}$ and the '$-$'

notation represents the trace taken from D and $\gamma \in L^\infty_+(\partial D)$.

The Problem Under Consideration

Therefore, the BVP for the electrostatic potential is given by

$$-\Delta u = 0 \text{ in } \Omega \backslash \partial D \quad \text{with} \quad u\big|_{\partial\Omega} = f \quad \text{and} \quad [\![\partial_\nu u]\!]\big|_{\partial D} = \gamma\, u$$

for any given $f \in H^{1/2}(\partial\Omega)$.

We also let $u_\emptyset \in H^1(\Omega)$ is the Harmonic Lifting of $f \in H^{1/2}(\partial\Omega)$.

Inverse Problem

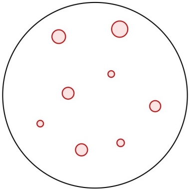
Reconstruct the support of the inclusion $D \subset \Omega$ from the knowledge of the Dirichlet-to-Neumann (DtN) maps from $H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$

$$\Lambda_D f = \partial_\nu u\big|_{\partial\Omega} \quad \text{and} \quad \Lambda_\emptyset f = \partial_\nu u_\emptyset\big|_{\partial\Omega}.$$

Recovering Small Targets

Asymptotic Analysis

Assume $0 < \epsilon \ll 1$ such that $D = \bigcup_{j=1}^{J} (x_j + \epsilon B_j)$ with $|B_j| = \mathcal{O}(1)$.



Using Green's 2nd theorem for $z \in \partial\Omega$ we have that

$$(\Lambda_D - \Lambda_\emptyset) f(z) = - \int_{\partial D} \gamma(x) u(x) \partial_{\nu(z)} \mathbb{G}(x,z) \, \mathrm{d}s(x)$$

where

$$\Delta \mathbb{G}(\cdot\,, z) = -\delta(\cdot - z) \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbb{G}(\cdot\,, z) = 0 \quad \text{on} \quad \partial\Omega.$$

Norm Estimate

We have that $\|u - u_{\emptyset}\|_{H^1(\Omega)} \leq C\,\epsilon^{\frac{d}{2}\left(1-\frac{2}{p}\right)}\|f\|_{H^{1/2}(\partial\Omega)}$ for the values $p \geq 2$ in $d = 2$ and $2 \leq p \leq 6$ in $d = 3$.

Norm Estimate

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From the estimate, we can conclude that for any $z \in \partial\Omega$ have that

$$\int_{\partial D} \gamma(x)(u - u_\emptyset)(x)\partial_{\nu(z)}\mathbb{G}(x,z)\,\mathrm{d}s(x) = \mathcal{O}\left(\epsilon^d\right) \quad \text{as} \quad \epsilon \to 0.$$

We can easily see that

$$(\Lambda_D - \Lambda_\emptyset)f(z) = -\int_{\partial D} \gamma(x)u_\emptyset(x)\partial_{\nu(z)}\mathbb{G}(x,z)\,\mathrm{d}s(x) - \int_{\partial D} \gamma(x)(u - u_\emptyset)(x)\partial_{\nu(z)}\mathbb{G}(x,z)\,\mathrm{d}s(x).$$

Asymptotic Expansion

For all $z \in \partial\Omega$ we have that as $\epsilon \to 0$

$$\int_{\partial D} \gamma(x) u_\emptyset(x) \partial_{\nu(z)} \mathbb{G}(x,z)\, \mathrm{d}s(x) = \epsilon^{d-1} \sum_{j=1}^{J} |\partial B_j| \mathrm{Avg}(\gamma_j) u_\emptyset(x_j) \partial_{\nu(z)} \mathbb{G}(x_j, z) + \mathcal{O}(\epsilon^d).$$

Asymptotic Expansion

For all $z \in \partial\Omega$ we have that as $\epsilon \to 0$

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$$\epsilon^{d-1} \sum_{j=1}^{J} |\partial B_j| \mathrm{Avg}(\gamma_j) u_\emptyset(x_j) \partial_{\nu(z)} \mathbb{G}(x_j, z) + \mathcal{O}(\epsilon^d).$$

From these asymptotic results we have that for any $z \in \partial\Omega$ we have that

$$(\Lambda_D - \Lambda_\emptyset) f(z) = -\epsilon^{d-1} \sum_{j=1}^{J} |\partial B_j| \mathrm{Avg}(\gamma_j) u_\emptyset(x_j) \partial_{\nu(z)} \mathbb{G}(x_j, z) + \mathcal{O}(\epsilon^d)$$

as the parameter $\epsilon \to 0$.

MUSIC Algorithm for $\Omega = B(0,1) \subset \mathbb{R}^2$

For any $f, g \in H^{1/2}(\partial\Omega)$, we have that

$$\left\langle g, \overline{(\Lambda_D - \Lambda_\emptyset) f} \right\rangle_{\partial\Omega} = \epsilon^{d-1} \sum_{j=1}^{J} |\partial B_j| \text{Avg}(\gamma_j) u_\emptyset(x_j; g) u_\emptyset(x_j; f) + \mathcal{O}(\epsilon^d)$$

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Define: matrix $\mathbf{F}_{n,m} = \epsilon^{d-1} \sum_{j=1}^{J} |\partial B_j| \text{Avg}(\gamma_j) u_\emptyset(x_j; e^{\text{i}m\theta}) u_\emptyset(x_j; e^{\text{i}n\theta})$

MUSIC Algorithm

Assume that $N + 1 > J$. Then for all $x \in \Omega$

$$\phi_x \in Range(\mathbf{F}\mathbf{F}^*) \quad \text{if and only if} \quad x \in \{x_j : j = 1, \cdots, J\}$$

where $\phi_x = \left[u_\emptyset \left(x; e^{\text{i}n\theta} \right) \right]_{n=0}^{N}$ for fixed $x \in \Omega$.

Let $\mathbf{P}$ be the orthogonal projection onto the Null($\mathbf{FF}^*$). Therefore,

$$\|\mathbf{P}\phi_x\| = 0 \quad \text{if and only if} \quad x \in \{x_j : j = 1, \cdots, J\}.$$

So the imaging functional is given by $W_{MUSIC}(x) = \|\mathbf{P}\phi_x\|^{-1}$.

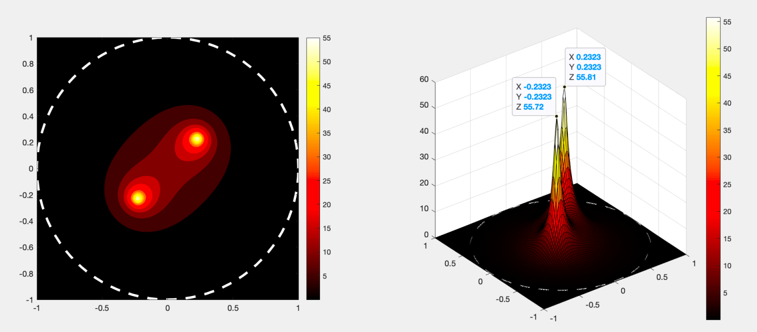
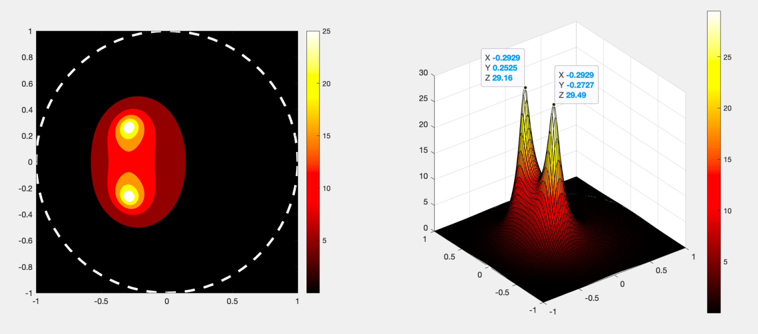


Figure: Reconstruction of to circles with $\epsilon = 0.01$ with centers $(0.25, 0.25)$ and $(-0.25, -0.25)$ where $\delta = 1\%$ noise is added to the data.



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Recovering Extended Targets

Initial Factorization of the DtN mappings

For a given $h \in L^2(\partial D)$, define $w \in H^1(\Omega)$ to be the unique solution to

$$-\Delta w = 0 \text{ in } \Omega \backslash \partial D \quad \text{with} \quad w\big|_{\partial\Omega} = 0 \quad \text{and} \quad [\![\partial_\nu w]\!]\big|_{\partial D} = \gamma\, h.$$

Define: the operator $G : L^2(\partial D) \to H^{-1/2}(\partial\Omega)$ by $Gh = \partial_\nu w\big|_{\partial\Omega}$

By well-posedness, $(\Lambda_D - \Lambda_\emptyset)f = \partial_\nu w\big|_{\partial\Omega}$ provided that $h = u\big|_{\partial D}$.

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By well-posedness, $(\Lambda_D - \Lambda_\emptyset)f = \partial_\nu w\big|_{\partial\Omega}$ provided that $h = u\big|_{\partial D}$.

Define: the operator $S : H^{1/2}(\partial\Omega) \to L^2(\partial D)$ by $Sf = u|_{\partial D}$.

From this we have the factorization $(\Lambda_D - \Lambda_\emptyset) = GS$.

Analysis of the Operator S

The operator S and it's adjoint

The adjoint operator $S^* : L^2(\partial D) \to H^{-1/2}(\partial\Omega)$ is given by

$$S^* g = \partial_\nu v\big|_{\partial\Omega}$$

where $v \in H^1(\Omega)$ satisfies

$$-\Delta v = 0 \;\text{ in }\; \Omega\backslash\partial D \quad \text{with} \quad v\big|_{\partial\Omega} = 0 \quad \text{and} \quad [\![\partial_\nu v]\!]\big|_{\partial D} = \gamma\, v + g.$$

Moreover, the operator S is compact and injective.

Question: Can factorize $(\Lambda_D - \Lambda_\emptyset) = S^* T S$ for some T?

Decomposing $G = S^* T$

Find $g \in L^2(\partial D)$ such that $S^* g = \partial_\nu w\big|_{\partial\Omega}$ where $w \in H^1(\Omega)$ solves:

$$-\Delta w = 0 \text{ in } \Omega\backslash\partial D \quad \text{with} \quad w\big|_{\partial\Omega} = 0 \quad \text{and} \quad [\![\partial_\nu w]\!]\big|_{\partial D} = \gamma\, h.$$

as well as

$$-\Delta w = 0 \text{ in } \Omega\backslash\partial D \quad \text{with} \quad w\big|_{\partial\Omega} = 0 \quad \text{and} \quad [\![\partial_\nu w]\!]\big|_{\partial D} = \gamma\, w + g.$$

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Therefore, $g = \gamma h - \gamma w \in L^2(\partial D)$ and we have $Gh = S^* g$.

Define: the operator $T : L^2(\partial D) \to L^2(\partial D)$ as $Th = \gamma(h - w)|_{\partial D}$.

Symmetric Factorization

From the operators we have previously defined we have that

$$(\Lambda_D - \Lambda_\emptyset) = S^* TS$$

and we have the following result.

Properties of the DtN mapping

The difference of the DtN mappings $(\Lambda_D - \Lambda_\emptyset) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is compact, injective, and has dense range.

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Properties of the DtN mapping

The difference of the DtN mappings $(\Lambda_D - \Lambda_\emptyset) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is compact, injective, and has dense range.

1. Compactness follows from the fact that S is compact
2. Prove the identity:

$$\langle f, (\Lambda_D - \Lambda_\emptyset) f \rangle_{\partial\Omega} = \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int_\Omega |\nabla u_\emptyset|^2 \, \mathrm{d}x + \int_{\partial D} \gamma |u|^2 \, \mathrm{d}s.$$

Theorem: Regularized Factorization Method

Assume that

$$A = S^* T S \quad \text{where} \quad S : X \to V \quad \text{and} \quad T : V \to V$$

such that:

- The operator S is compact and injective.
- The operator T is strictly coercive on Range(S).

Then we have that

$$\ell \in \text{Range}(S^*) \quad \Longleftrightarrow \quad \liminf_{\alpha \to 0} \langle x_\alpha \, , A x_\alpha \rangle_{X \times X^*} < \infty$$

where x_α is the regularized solution to $Ax = \ell$.

I. Harris, Regularization of the Factorization Method applied to diffuse optical tomography. *Inverse Problems* **37** 125010 (2021) arXiv:2106.07743.

Range of S^*

We have that $\partial_\nu \mathbb{G}(\cdot\,,z)\big|_{\partial\Omega} \in \text{Range}(S^*)$ if and only if $z \in D$.

Here $\mathbb{G}(\cdot\,,z)$ is the solution to the problem

$$\Delta \mathbb{G}(\cdot\,,z) = -\delta(\cdot - z) \quad \text{in} \quad \Omega \quad \text{and} \quad \mathbb{G}(\cdot\,,z) = 0 \quad \text{on} \quad \partial\Omega.$$

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Coercivity of T

The operator $T : L^2(\partial D) \to L^2(\partial D)$ given by $Th = \gamma(h - w)\big|_{\partial D}$ is strictly coercive on $L^2(\partial D)$.

By the BVP for w equation and Green's 1st Theorem we have that

$$(Th, h)_{L^2(\partial D)} = \int_{\partial D} \gamma |h|^2 \, \mathrm{d}s + \int_{\Omega} |\nabla w|^2 \, \mathrm{d}x.$$

Characterization of the region D

The Main Result

The mappings $(\Lambda_D - \Lambda_\emptyset) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ uniquely determines D such that for any $z \in \Omega$

$$z \in D \quad \text{if and only if} \quad \liminf_{\alpha \to 0} \langle f_\alpha^z, (\Lambda_D - \Lambda_\emptyset) f_\alpha^z \rangle_{\partial\Omega} < \infty$$

where f_α^z is the regularized solution to $(\Lambda_D - \Lambda_\emptyset) f^z = \partial_\nu \mathbb{G}(\cdot\,, z)\big|_{\partial\Omega}$.

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We can show that

$$\Big\langle f_\alpha^z\,, (\Lambda_D - \Lambda_\emptyset) f_\alpha^z \Big\rangle_{\partial\Omega} = \sum \frac{\Phi_\alpha^2(\lambda_n)}{\lambda_n} \Big| \langle \psi_n, \partial_\nu \mathbb{G}(\cdot\,, z) \rangle_{\partial\Omega} \Big|^2$$

with $\{\lambda_n; \psi_n\} \in \mathbb{R}_{>0} \times H^{1/2}(\partial\Omega)$ the singular values and right singular vector of the DtN mapping $(\Lambda_D - \Lambda_\emptyset)$.

Reconstruction with $\alpha = 10^{-7}$ and transmission parameter $\gamma = 1$

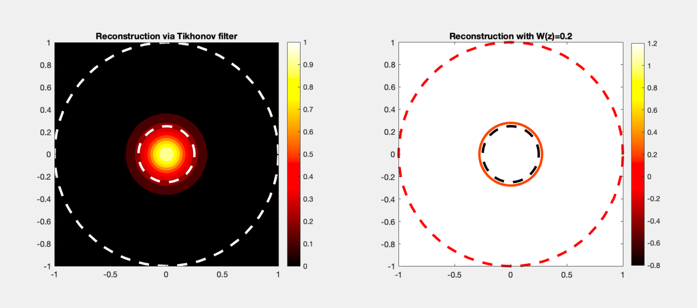

Figure: Reconstruction of a circular domain with noise level $\delta = 1\%$.

Here the DtN mapping is computed via separation of variables.

Reconstruction with $\alpha = 10^{-5}$ and transmission parameter

$$\gamma(x(\theta)) = \frac{1}{4 + \exp(\cos\theta)}$$

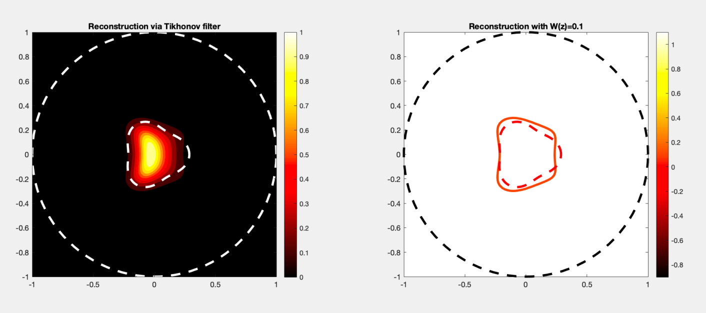

Figure: Reconstruction of a pear-shaped domain with noise level $\delta = 2\%$

Here the DtN mapping is computed via a spectral method.

The Regularization Step is Necessary

An Inverse Scattering Problem: we let $D \subset \mathbb{R}^d$ with $u^s \in H^1_{loc}(\mathbb{R}^d)$

$$\Delta u^s + k^2(1+q)u^s = -k^2 q e^{ikx\cdot\hat{y}} \quad \text{in} \quad \mathbb{R}^d \qquad + \text{SRC as } |x| \to \infty.$$

I. Harris, Regularization of the Factorization Method with Applications to Inverse Scattering. *Accepted* AMS Contemporary Mathematics (arXiv:2202.13411).

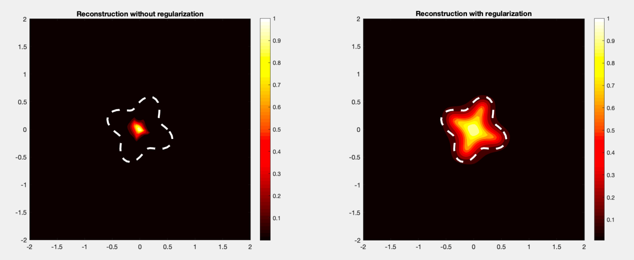

Figure: Left: reconstruction without regularization and Right: reconstruction with regularization. Here 10% error is added to the far-field data.

Some Ongoing Work

1) Using the Reciprocity Gap Functional along with asymptotic expansions to recover $D = \bigcup_{j=1}^{J} (x_j + \epsilon B_j)$ from the Cauchy data from:

$$-\Delta u + \chi(D)u = 0 \;\text{ in }\; \Omega \quad \text{with} \quad u = f \;\text{ on }\; \partial\Omega$$

and

$$\Delta u^s + k^2 u^s = \chi(D) \;\text{ in }\; \Omega \quad \text{with} \quad u^s = f \;\text{ on }\; \partial\Omega.$$

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and

$$\Delta u^s + k^2 u^s = \chi(D) \ \text{ in } \ \Omega \quad \text{with} \quad u^s = f \ \text{ on } \ \partial\Omega.$$

2) Consider the case $A^\delta : X \to X^*$ such that $\|A^\delta - A\| \to 0$ as $\delta \to 0$ then

$$\ell \in \text{Range}(S^*) \quad \Longleftrightarrow \quad \liminf_{\alpha \to 0} \liminf_{\delta \to 0} \langle x_\alpha^\delta, A^\delta x_\alpha^\delta \rangle_{X \times X^*} < \infty$$

where x_α^δ is the regularized solution to $A^\delta x = \ell$.


"That's all folks!"