A Theory of Strongly Coupled Oscillators

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SIAM + Applied & Numerical Analysis Seminar

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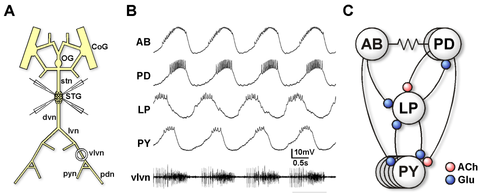
Coupled Oscillators Example 1: Stomatogastric Ganglion

Crab stomatogastric ganglion (Marder lab, Brandeis U.)



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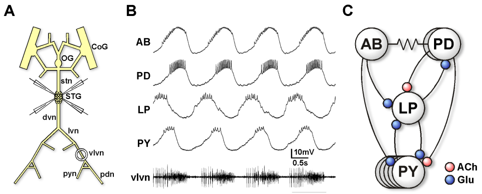

- Computationally complex

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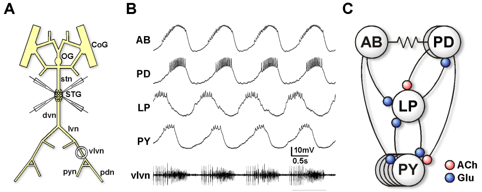

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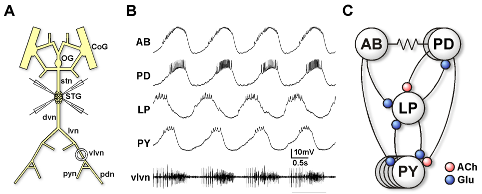

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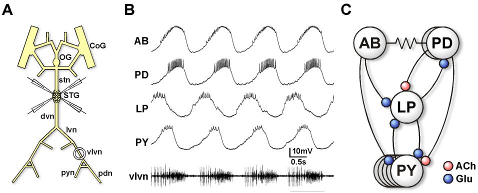

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Coupled Oscillators Example 2: Chemical Oscillators

Norton et al. PRL, 2019 (Fraden Lab, Brandeis U.)

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- How can we understand the existence and stability of phase-locked solutions?

Limit Cycles

$$\dot{x} = F(x)$$

- $F : \mathbb{R}^n \to \mathbb{R}^n$

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$$\dot{\mathsf{x}} = \mathsf{F}(\mathsf{x})$$

- $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$
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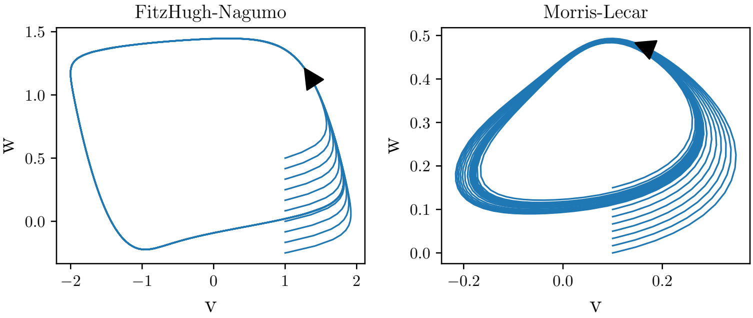

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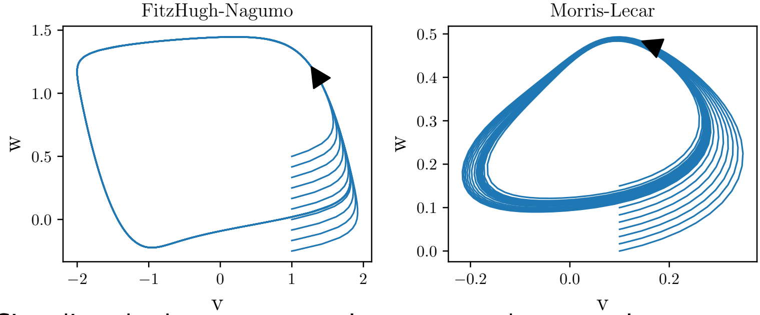
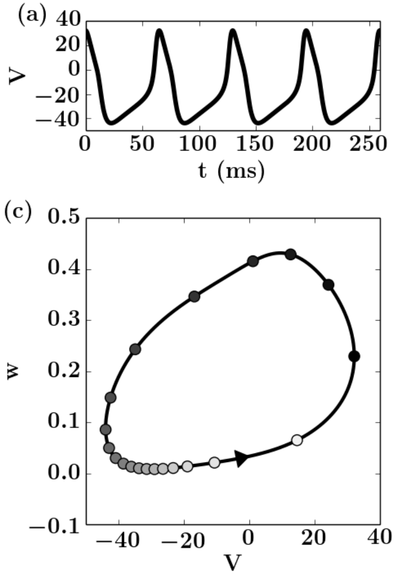


Circadian rhythms, gene regulatory networks, central pattern generators, Belousov-Zhabotinsky, ...

Reduce Each Oscillator to Phase Angle



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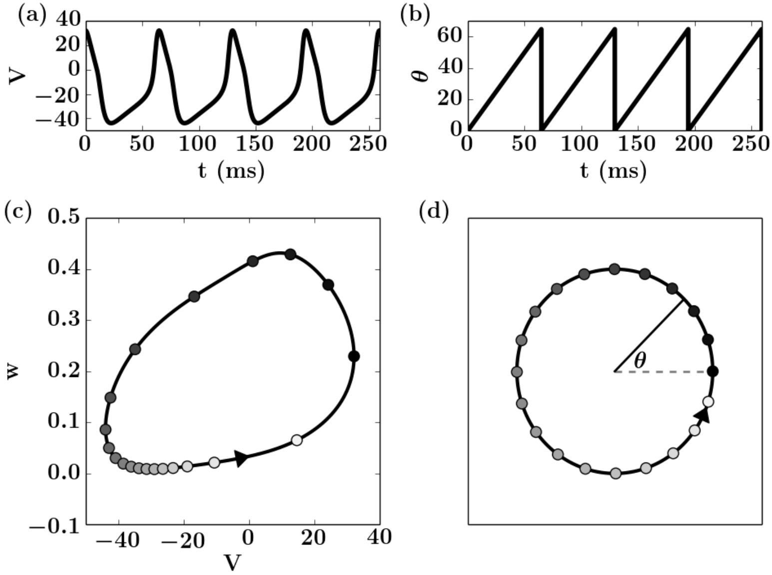

Weak Coupling Theory

$$\dot{x}_i = F(x_i) + \varepsilon G(x_i, x_{3-i}), \quad x_i \in \mathbb{R}^n, \quad i = 1, 2,$$

where G is a coupling term (e.g., diffusion, chemical synapse, gap junction).

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$$\dot{\theta}_i = 1 + \varepsilon (\text{Phase Response}) * G(x_i, x_{3-i}), \quad \theta_i \in \mathbb{R}.$$

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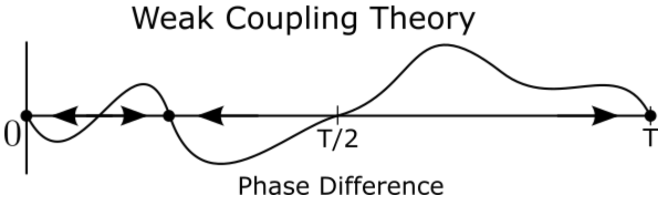
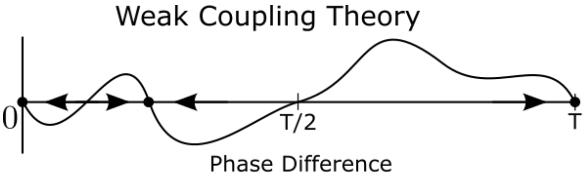
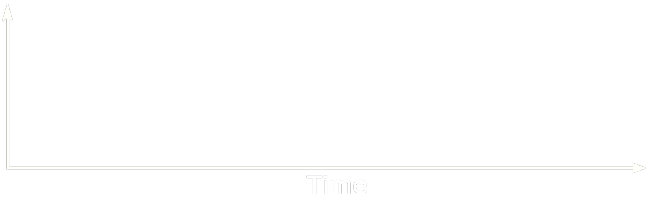
Cartoon of the right-hand side of $\dot{\phi}$

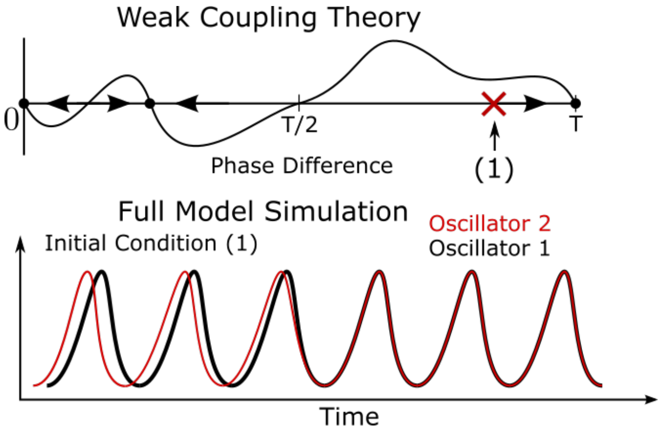


Weak Coupling Theory: Phase Difference Equation

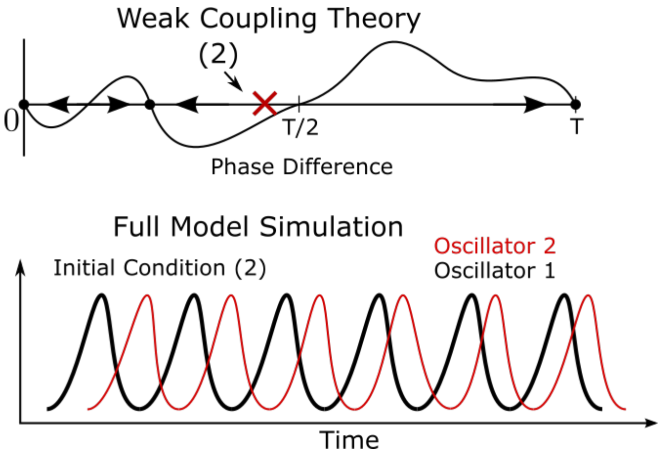




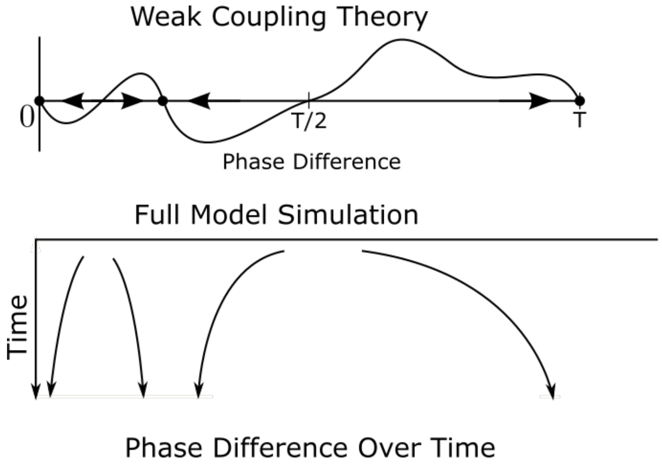
Weak Coupling Theory: Initial Condition 1



Weak Coupling Theory: Initial Condition 2

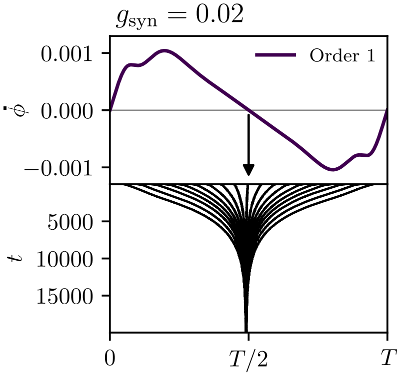


Weak Coupling Theory: Phase Differences Over Time



What Happens for Stronger Coupling?

Thalamic neural model (2×4 dimensions) with chemical synaptic coupling ($g_{\text{syn}} = \varepsilon$)



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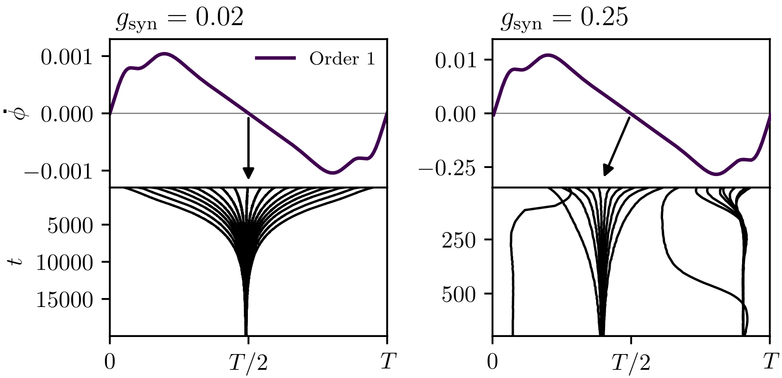
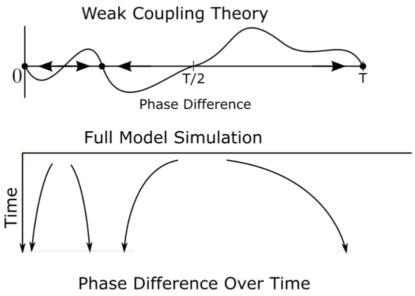


Takeaways from Weakly Coupled Oscillator Theory



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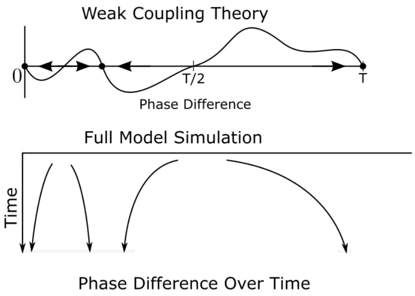

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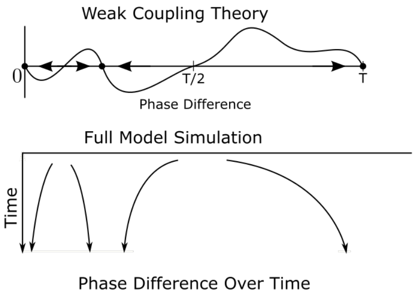

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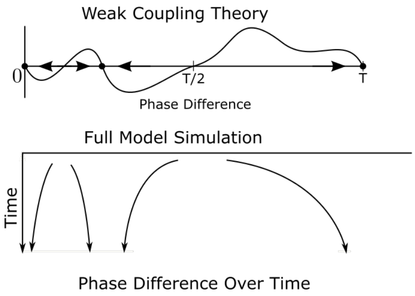

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- Fixed points capture long-term phase-locking behavior.
- Generalizable to N oscillators (N algebraic equations in N unknowns).
- **Want these same benefits for stronger coupling**.

Derivation of Coupling Functions: Assumptions

$$\dot{x}_1 = F(x_1) + \varepsilon G(x_1, x_2),$$
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- $|\varepsilon| \geq 0$ not necessarily small.
- $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, general smooth coupling function.

Isostable Reduction: Change of Coordinates

Transform the system into phase θ_i.

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- In non-weak coupling theory, $\nabla\theta_i = \mathcal{Z}(\theta_i, \psi_i)$.

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Transform the system into phase θ_i.

$$\begin{aligned}\frac{d\theta_1}{dt} &= \nabla\theta_1 \cdot \frac{dx_1}{dt} \\ &= \nabla\theta_1 \cdot [F(x_1) + \varepsilon G(x_1, x_2)] \\ &= \nabla\theta_1 \cdot F(x_1) + \varepsilon\nabla\theta_1 \cdot G(x_1, x_2) \\ &= 1 + \varepsilon\nabla\theta_1 \cdot G(x_1, x_2)\end{aligned}$$

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  - ψ_i is an amplitude coordinate.

Isostable Reduction: Change of Coordinates in ψ_1

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- κ Floquet exponent.
- $\nabla\psi_i \equiv \mathcal{I}(\theta_i, \psi_i)$ quantifies amplitude shifts

Isostable Reduction: Change of Coordinates

[Park and Wilson, 2021]

$$\dot{\theta}_i = 1 + \varepsilon \mathcal{Z}(\theta_i, \psi_i) \cdot \mathsf{G}(\theta_i, \psi_i, \theta_j, \psi_j),$$
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- θ_i phase, ψ_i amplitude.

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- Next step: reduce these 4 equations into 2 equations.

Higher-Order Coupling Functions

Expand everything in ψ, ε:

$$\mathcal{Z}(\theta_i, \psi_i) \approx \mathsf{Z}^{(0)}(\theta_i) + \psi_i \mathsf{Z}^{(1)}(\theta_i) + \psi_i^2 \mathsf{Z}^{(2)}(\theta_i) + \ldots,$$

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\mathcal{I}(\theta_i, \psi_i) &\approx \mathsf{I}^{(0)}(\theta_i) + \psi_i \mathsf{I}^{(1)}(\theta_i) + \psi_i^2 \mathsf{I}^{(2)}(\theta_i) + \dots, \\
\mathsf{x}_i(t) &\approx \mathsf{Y}(\theta_i) + \psi_i \mathsf{g}^{(1)}(\theta_i) + \psi_i^2 \mathsf{g}^{(2)}(\theta_i) + \dots,
\end{aligned}$$

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\mathsf{x}_i(t) &\approx \mathsf{Y}(\theta_i) + \psi_i \mathsf{g}^{(1)}(\theta_i) + \psi_i^2 \mathsf{g}^{(2)}(\theta_i) + \dots, \\
\psi_i(t) &\approx \varepsilon p_i^{(1)}(t) + \varepsilon^2 p_i^{(2)}(t) + \varepsilon^3 p_i^{(3)}(t) + \dots.
\end{aligned}$$

Higher-Order Coupling Functions

Expand everything in ψ, ε:

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- Put into the phase-amplitude equations

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- Put into the phase-amplitude equations
- Combinatorial explosion of terms.

Higher-Order Coupling Functions

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\end{aligned}$$

- Put into the phase-amplitude equations
- Combinatorial explosion of terms.
- Use symbolic packages to collect terms.

Higher-Order Coupling Functions

Eliminate the amplitude equations

$$\dot{\theta}_i = 1 + \varepsilon \mathcal{Z}(\theta_i, \psi_i) \cdot \mathsf{G}(\theta_i, \psi_i, \theta_j, \psi_j),$$
$$\dot{\psi}_i = \kappa \psi_i + \varepsilon \mathcal{I}(\theta_i, \psi_i) \cdot \mathsf{G}(\theta_i, \psi_i, \theta_j, \psi_j).$$

- Each term in the expansion $\psi_i(t) \approx \varepsilon p_i^{(1)}(t) + \varepsilon^2 p_i^{(2)}(t) + \varepsilon^3 p_i^{(3)}(t) + \ldots$ satisfies a linear ODE.

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- Solve for each $p_i^{(k)}$ in terms of $p^{(k-1)}$ terms or lower.

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- Now we have 2 equations, one for each θ_i.

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- Plug back into the equation for θ_i.
- Now we have 2 equations, one for each θ_i.

$$\dot{\theta}_i = 1 + \varepsilon[\mathsf{Z}^{(0)}(\theta_i) + \psi_i \mathsf{Z}^{(1)}(\theta_i) + \psi_i \mathsf{Z}^{(2)}(\theta_i) + \ldots] \cdot \mathsf{G}(\theta_i, \psi_i, \theta_j, \psi_j),$$

where $\psi_i(t) \approx \varepsilon p_i^{(1)}(t) + \varepsilon^2 p_i^{(2)}(t) + \varepsilon^3 p_i^{(3)}(t) + \ldots$.

Higher-Order Coupling Functions

Finally, collect in powers of ε and take the average

$$\dot{\theta}_1 = \varepsilon\mathcal{H}^{(1)}(\theta_2 - \theta_1) + \varepsilon^2\mathcal{H}^{(2)}(\theta_2 - \theta_1) + \varepsilon^3\mathcal{H}^{(3)}(\theta_2 - \theta_1) + \dots,$$

where

$$\mathcal{H}^{(1)}(\theta) = \frac{1}{T}\int_0^T Z^{(0)} \cdot M^{(0,0)} ds.$$

$$\mathcal{H}^{(2)}(\theta) = \frac{1}{T}\int_0^T p_1^{(1)} Z^{(1)} \cdot M^{(0,0)} + p_2^{(1)} Z^{(0)} \cdot M^{(0,1)} + p_1^{(1)} Z^{(0)} M^{(1,0)} ds.$$

$$\mathcal{H}^{(3)}(\theta) = \frac{1}{T}\int_0^T \left[p_1^{(2)} Z^{(1)} \cdot M^{(0,0)} + \left(p_1^{(1)}\right)^2 Z^{(2)} \cdot M^{(0,0)} \right.$$
$$\left. + p_1^{(1)} p_2^{(1)} Z^{(1)} \cdot M^{(0,1)} + \left(p_1^{(1)}\right)^2 Z^{(1)} \cdot M^{(1,0)}\right] ds.$$

Higher-Order Coupling Functions

Finally, collect in powers of ε and take the average

$$\dot{\theta}_1 = \varepsilon \mathcal{H}^{(1)}(\theta_2 - \theta_1) + \varepsilon^2 \mathcal{H}^{(2)}(\theta_2 - \theta_1) + \varepsilon^3 \mathcal{H}^{(3)}(\theta_2 - \theta_1) + \ldots,$$

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$$\mathcal{H}^{(3)}(\theta) = \frac{1}{T}\int_0^T \Big[p_1^{(2)} Z^{(1)} \cdot M^{(0,0)} + \left(p_1^{(1)}\right)^2 Z^{(2)} \cdot M^{(0,0)} + p_1^{(1)} p_2^{(1)} Z^{(1)} \cdot M^{(0,1)} + \left(p_1^{(1)}\right)^2 Z^{(1)} \cdot M^{(1,0)} \Big] ds.$$

- $M^{(i,j)}$ are partial derivatives of the coupling function G.

Higher-Order Coupling Functions

Finally, collect in powers of ε and take the average

$$\dot{\theta}_1 = \varepsilon\mathcal{H}^{(1)}(\theta_2-\theta_1) + \varepsilon^2\mathcal{H}^{(2)}(\theta_2-\theta_1) + \varepsilon^3\mathcal{H}^{(3)}(\theta_2-\theta_1) + \ldots,$$

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$$\mathcal{H}^{(3)}(\theta) = \frac{1}{T}\int_0^T \Big[p_1^{(2)} Z^{(1)} \cdot M^{(0,0)} + \left(p_1^{(1)}\right)^2 Z^{(2)} \cdot M^{(0,0)} + p_1^{(1)} p_2^{(1)} Z^{(1)} \cdot M^{(0,1)} + \left(p_1^{(1)}\right)^2 Z^{(1)} \cdot M^{(1,0)} \Big] ds.$$

- $M^{(i,j)}$ are partial derivatives of the coupling function G.
- $Z^{(k)} = Z^{(k)}(s)$, $M^{(k,\ell)} = M^{(k,\ell)}(s, \theta + s)$,
  $p_1^{(k)} = p_1^{(k)}(s, \theta+s)$, $p_2^{(k)} = p_2^{(k)}(\theta+s, s)$.

Higher-Order Coupling Functions

$$\dot{\theta}_1 = \varepsilon\mathcal{H}^{(1)}(\theta_2 - \theta_1) + \varepsilon^2\mathcal{H}^{(2)}(\theta_2 - \theta_1) + \varepsilon^3\mathcal{H}^{(3)}(\theta_2 - \theta_1) + \dots,$$

- **Caveat**: used first-order averaging theory:

$$\dot{\mathsf{x}} = \varepsilon \mathsf{F}(\mathsf{x}, t, \varepsilon), \quad \mathsf{x}(0) = \mathsf{x}_0,$$

F is T-periodic in t. Consider the averaged equation,

$$\dot{\mathsf{z}} = \varepsilon \overline{\mathsf{F}}(\mathsf{z}), \quad \mathsf{z}(0) = \mathsf{z}_0,$$

where $\bar{\mathsf{F}} = \frac{1}{T}\int_0^T \mathsf{F}(\mathsf{x}, s, 0)ds$. Then $\mathsf{x}(t) = \mathsf{z}(t) + O(\varepsilon)$ for $O(1/\varepsilon)$ time.

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- In practice, first-order averaging is sufficient.

Phase Difference Dynamics

Let $\phi = \theta_2 - \theta_1$.

$$\begin{aligned}\dot{\phi} = \varepsilon \left[\mathcal{H}^{(1)}(-\phi) - \mathcal{H}^{(1)}(\phi)\right] \\ + \varepsilon^2 \left[\mathcal{H}^{(2)}(-\phi) - \mathcal{H}^{(2)}(\phi)\right] \\ + \varepsilon^3 \left[\mathcal{H}^{(3)}(-\phi) - \mathcal{H}^{(3)}(\phi)\right] + \dots,\end{aligned}$$

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- $\mathcal{H}^{(3)}(-\phi) - \mathcal{H}^{(3)}(\phi)$ **and beyond**: [Park and Wilson, 2021]

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Takeaway: A scalar ODE describes the phase difference dynamics for ε not necessarily small.

Application to a "Simple" Model

Complex Ginzburg-Landau

$$
\begin{aligned}
x_j' &= (1 - x_j^2 - y_j^2)x_j - q(x_j^2 + y_j^2)y_j + \varepsilon\left[x_k - x_j - d(y_k - y_j)\right], \\
y_j' &= (1 - x_j^2 - y_j^2)y_j + q(x_j^2 + y_j^2)x_j + \varepsilon\left[y_k - y_j + d(x_k - x_j)\right],
\end{aligned}
$$

$j = 3 - k$ with $k = 1, 2$.

- Tractable model.

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- For $d = 0.4, \varepsilon = 0.3$, antiphase is stable.

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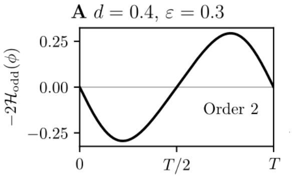

Application to a "Simple" Model

Complex Ginzburg-Landau

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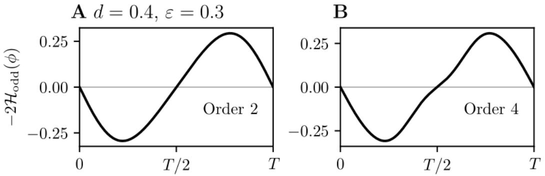

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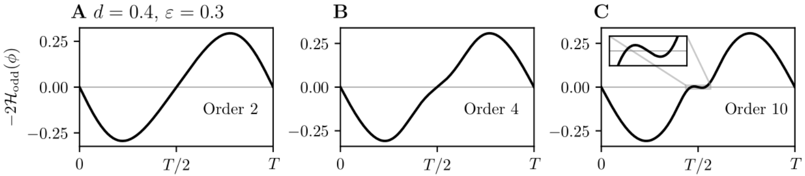
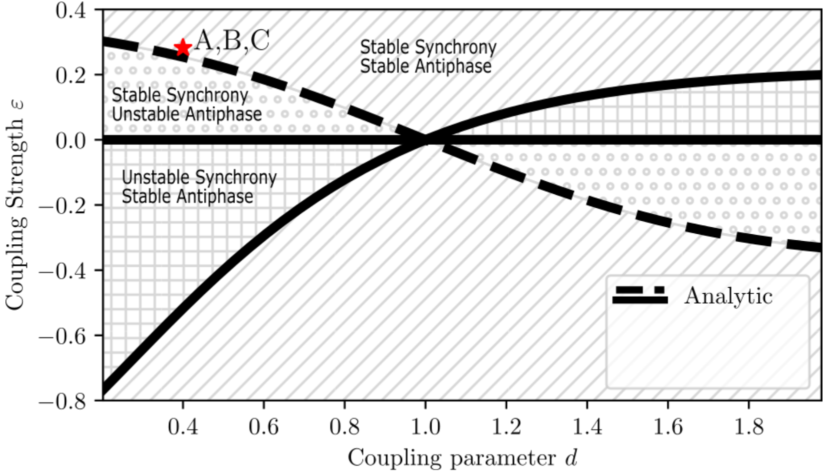


Application to a “Simple” Model: Two Parameter Diagram



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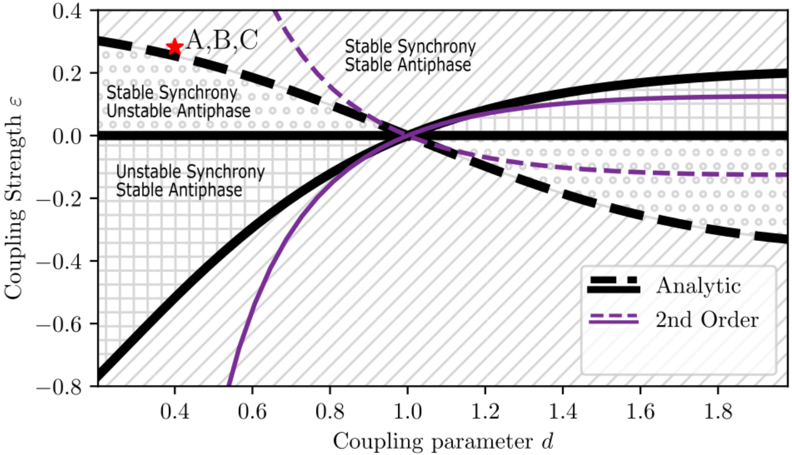

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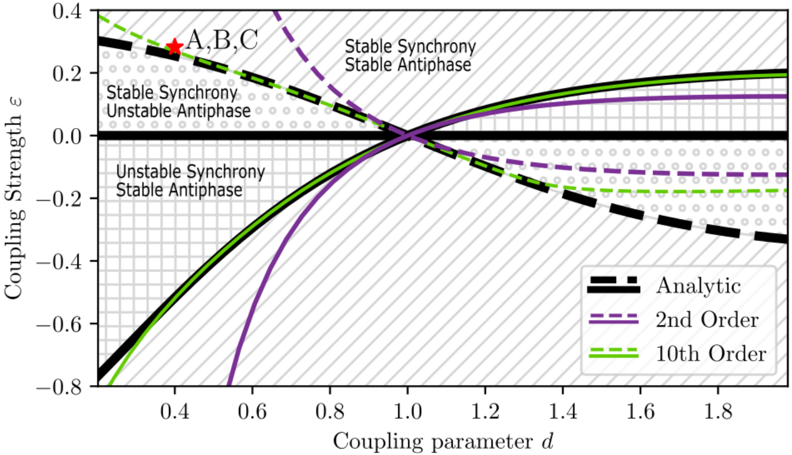

Application to a Neural Model

Thalamic neuron model

$$C\frac{dV_i}{dt} = -I_{\mathrm{L}}(V_i) + I_{\mathrm{Na}}(V_i) + I_{\mathrm{K}}(V_i) + I_{\mathrm{T}}(V_i) - g_{\mathrm{syn}}w_j(V_i - E_{\mathrm{syn}}) + I_{\mathrm{app}},$$
$$\frac{dh_i}{dt} = (h_\infty(V_i) - h_i)/\tau_h(V_i),$$
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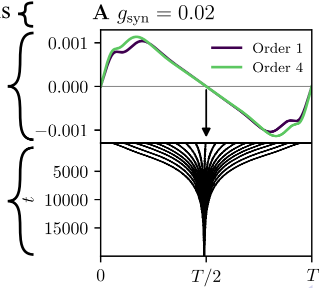
Coupling Strengths {

Coupling Theory

Phase differences in the full model



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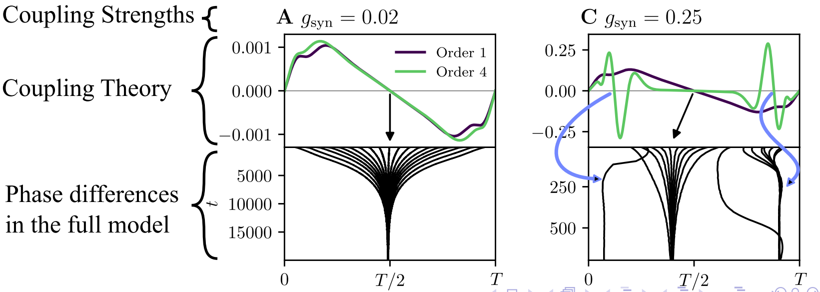
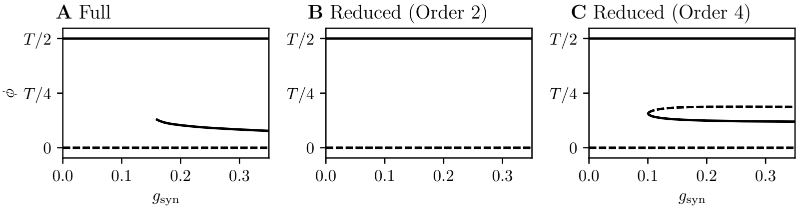


Bifurcation Diagram in Phase-Locked States



Summary and Future Directions

- Derived interaction functions $\mathcal{H}^{(i)}$ in the case of strong coupling.

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- Caveat of first-order averaging theory.

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Two-Oscillator Isostable Reduction: Change of Coordinates

$$\begin{aligned}\frac{d\theta_i}{dt} &= \nabla\theta_i \cdot \frac{d\mathsf{x}_1}{dt} \\ &= \nabla\theta_i \cdot [\mathsf{F}(\mathsf{x}_1) + \varepsilon \mathsf{G}(\mathsf{x}_1, \mathsf{x}_2)] \\ &= \nabla\theta_i \cdot \mathsf{F}(\mathsf{x}_1) + \varepsilon \nabla\theta_i \cdot \mathsf{G}(\mathsf{x}_1, \mathsf{x}_2) \\ &= 1 + \varepsilon \nabla\theta_i \cdot \mathsf{G}(\mathsf{x}_1, \mathsf{x}_2)\end{aligned}$$

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N-Oscillator Isostable Reduction

$$\dot{\theta}_i = 1 + \varepsilon \mathcal{Z}(\theta_i, \psi_i) \cdot \sum_{j=1}^{N} a_{ij} G(\theta_i, \psi_i, \theta_j, \psi_j),$$

$$\dot{\psi}_i = \kappa \psi_i + \varepsilon \mathcal{I}(\theta_i, \psi_i) \cdot \sum_{j=1}^{N} a_{ij} G(\theta_i, \psi_i, \theta_j, \psi_j).$$

- θ_i phase, ψ_i amplitude.
- κ slowest decaying Floquet multiplier.
- $\mathcal{Z}$ general phase response.
- $\mathcal{I}$ general isostable response.